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Type-2 Fuzzy Sets for Modeling Uncertainty in Measurement

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Abstract – *A correct representation of uncertainty in measurement is crucial in many applications. Statistical approach sometimes is not the best choice, especially when the knowledge of the measurement process refers only to the support of the values and does not allow a correct assumption on the probability density function (pdf) of the measured variable. In this paper we present an approach that uses the concept of generalized fuzzy numbers, namely Type-2 fuzzy sets, in order to handle the intrinsic dispersion of the possible pdfs associated to a variable. The relation between our representation and the so called Random Fuzzy Variables (RFV) will be also investigated. The use of this representation allows to easily implement the uncertainty propagation, through a functional model, by working directly on the Type-2 fuzzy numbers and by evaluating simultaneously the propagation results for the whole set of confidence levels. Anyway, when a statistical analysis can be performed, the results can be embedded in this generalized representation. Moreover, the new approach allows to assign to the final measurement value a reliable confidence level also in this case, by combining the expanded uncertainty evaluated following IEC-ISO Guide recommendations with the Type-2 fuzzy numbers associated to the output variable. An example of this representation will be also provided.*

Keywords – *Uncertainty, probability-possibility transformations, Type-2 fuzzy variables.*

I. INTRODUCTION

The correct representation of the measurement associated to a given variable is a focal point in many applications. An exhaustive description of the principal recommendations about how a reliable expression of the measurement and of its uncertainty has to be performed, is contained in the IEC-ISO "Guide to the expression in measurement" [1] which we address almost totally. Principally, the IEC-ISO Guide states that the measurement cannot be expressed by a single value, but by a distribution of values over an interval within which the measurements lie with a given confidence level. So, detailed rules are provided in order to evaluate this distribution with the highest confidence level associated. The probabilistic approach represents the natural way of computing uncertainty estimation and performing uncertainty propagation through a functional model, but recently many limitations of this approach have been focused. In particular, in order to perform a correct probabilistic representation of the measurement, a set of independent observations is needed. However, in many applications, the value assigned to a certain variable is taken from manuals, calibration reports, handbook, reference values, so that any assumption on the probability density function (pdf) associated to a variable cannot be reliable. Moreover, in order to propagate the uncertainty through a generic function f, the joint pdf and the statistical correlation have to be estimated. Again, if a very weak knowledge is available about some of the involved variables, all these estimations can lead to a strong error propagation, thus producing eventually a biased expression of the combined uncertainty. Finally, in a forthcoming paper [2], it will be also highlighted that, even if the marginal pdf are known for each variable and the correlation coefficient is correctly estimated, then the choice of the joint pdf is not unique, thus causing a non univocity of the problem. In a recent literature [3], the authors also underline the fact that in many applications, the random effects do not prevailing over systematic ones; so, especially when systematic effects are unknown and the same for the corresponding correction factor, a probabilistic approach may yield a wrong evaluation of the measurement uncertainty.

In all these cases, in particular when a type-B evaluation [1] of uncertainty is needed, alternative methods have to be implemented.

Recently [4, 5], a fuzzy approach has been investigated in order to represent uncertainty in measurement when the available information is poor and does not allow a statistical analysis for uncertainty handling. The concepts of fuzzy variables and fuzzy sets have been introduced by Zadeh [6,7] as an extension of the traditional concept of membership of a variable a to a set A. In crisp set theory this membership is represented by a one ($a \in A$) or by a zero ($a \notin A$), whereas in fuzzy set theory it can be modeled by a MF $\mu_A(a)$ such that $0 \leqslant \mu_A(a) \leqslant 1$, with $\mu_A(a)$ convex and normal (i.e., there exists at least one value b such that $\mu_A(b) = 1$).

The set A is called *fuzzy subset* and the *support* of A is the set of points at which $\mu_A(a)$ is positive. The $\alpha-$level set (or $\alpha-$cut) of A is a non fuzzy set, denoted by $\mathcal{A}_\alpha$, defined as

$$\mathcal{A}_\alpha = \{a | \mu_A(a) \geqslant \alpha\}.$$

In [8] Zadeh also introduced the concept of possibility theory as a mathematical counterpart of probability theory that deals with uncertainty by means of fuzzy sets, so that a fuzzy/possibility approach is denoted. Moreover, in [4, 5] the authors also underline that the fuzzy/possibility approach is between interval analysis and probability theory. The former is the less expressive because uses only the information of upper and lower bounds of an interval, without any relation with a level of confidence (so with a membership degree). This is not sufficient to use IEC-ISO Guide recommendations in uncertainty expression. Otherwise, the probability approach is somehow too rich for representing relative lack of information coming from human experts or imprecise sensors.

So, in a recent literature [9] a probabilistic model is turned into possibilistic model, by means of a fuzzy set representation. This approach allows us to give a complete characterization of the uncertainty in measurement, as recommended in [1], adding that information intrinsically lacking, where neither a statistical nor a probabilistic analysis can be performed.

However, in many cases, a simple fuzzy set is not sufficient to express the incomplete knowledge in a certain measurement process. For example, this can be due to systematic errors prevailing on random effects, or to a natural dispersion of the possible pdfs associated to a certain variable. In the first case, the use of Random Fuzzy Variables (RFV) [3, 10, 11] has been widely investigated, whereas in the second case no innovative approach has been still proposed. For this reason, in this paper, we will inspect a generalized class of fuzzy numbers, namely Type-2 fuzzy numbers, firstly introduced by Zadeh [12], in order to easily express and propagate uncertainty in many practical cases. The approach that will be investigated, can be eventually integrated in standard methods (coming from type-A evaluation [1]) thus providing a reliable confidence level associated to the measurement when the incomplete knowledge represents a very critical aspect.

This new class of fuzzy sets are strictly related to RFV, as highlighted in [13] and this interesting relation will be also inspected. Finally, simulation results will be provided.

II. TYPE-2 FUZZY SETS FOR UNCERTAINTY HANDLING

In [12] Zadeh firstly introduced the concept of generalized fuzzy sets. Suppose that A is a fuzzy set and suppose that the MF $\mu_A(a)$ associated is allowed to be a fuzzy subset in the interval $[0, 1]$. In order to differentiate this kind of generalized fuzzy sets from the classical ones Zadeh refers to them as Type-2 fuzzy sets. More generally, he gives a recursive definition of Type-n fuzzy sets as follows

Definition II.1: A fuzzy set is of Type-n, $n = 2, 3, \ldots$, if its MF ranges over fuzzy sets of Type-$(n-1)$. The MF of a fuzzy set of Type-1 ranges over the interval $[0, 1]$.

In a recent literature [14, 15], various classes of Type-2 MFs are inspected, but anyway a particular one, including the so called Interval Type-2 fuzzy sets, has been widely investigated and applied in various contexts such as decision making, time-series forecasting, control of mobile robots [15], etc. Interval Type-2 fuzzy sets are the most widely used Type-2 fuzzy sets because they are simple to use and because it is very difficult to justify the use of any kind of Type-2 fuzzy sets. In this case, the MF $\mu_A(a)$ is an Interval Type-2 fuzzy set so that it can be represented only by its lower and upper bounds (i.e. by two Type-1 MFs). This situation is depicted in Fig. 1 and compared with other typologies of non Interval Type-2 MFs (denoted as General Type-2 MFs).

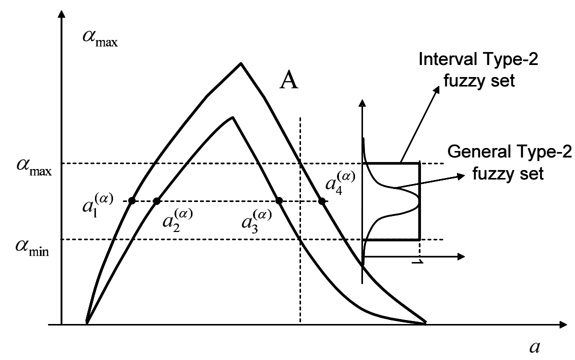


Fig. 1. Example of an interval Type-2 MF

In order to identify how to easily operate on this class of more complex fuzzy sets, in [13] the concept of interval of confidence of Type-2 is introduced. Let us recall now some basic notions. Assume that the lower and upper bounds of an interval of confidence, instead of being ordinary numbers, are fuzzy numbers, that themselves have intervals of confidence. We will denote this kind of Type-2 interval of confidence as

$$\mathcal{A} = [[a_1, a_2], [a_3, a_4]]\,,$$

such that $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$. When $a_1 = a_2$ and $a_3 = a_4$ the interval of confidence of Type-2 becomes an interval of Type-1 and if $a_1 = a_2 = a_3 = a_4$ the interval becomes of Type-0 (i.e., a number). Consider now a sequence of intervals of confidence of Type-2 that depends on α, that is

$$\forall \alpha \in [0, 1], \quad \forall a_1^{(\alpha)},\, a_2^{(\alpha)},\, a_3^{(\alpha)},\, a_4^{(\alpha)}$$

$$\mathcal{A}_\alpha = \left[[a_1^{(\alpha)}, a_2^{(\alpha)}], [a_3^{(\alpha)}, a_4^{(\alpha)}]\right],$$

such that $a_1^{(\alpha)} \leqslant a_2^{(\alpha)} \leqslant a_3^{(\alpha)} \leqslant a_4^{(\alpha)}$.

In order to perform algebraic operations on Type-2 fuzzy sets let us consider now that a fuzzy number of Type-2 can be constructed in two ways.

1) Given a Type-1 fuzzy number A and a convex fuzzy subset B we build a Type-2 fuzzy number as shown in Fig. 2 (a). Note that we can identify a gamma of Type-1 MFs belonging to the range $[B, A]$, as for example the dotted MF.
2) The second kind of construction considers a Type-1 fuzzy number A and its translation of a certain Δa thus obtaining Fig. 2 (b).

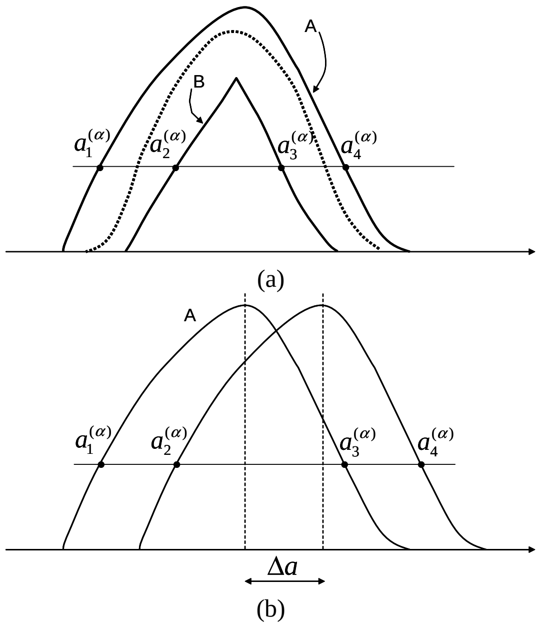


Fig. 2. Two ways of building a Type-2 fuzzy number (a) and (b)

The latter interpretation is commonly used in literature [14, 15]. It can be seen as a blurring of a Type-1 MF around a central value, thus producing the corresponding Type-2 MF. Otherwise, the former representation is the one we address in this paper, since the fuzzy subset B is naturally the inner MF (i.e., a lower bound) and the fuzzy set A corresponds to the outer MF (i.e., an upper bound). This point of view will allow us to directly construct the Type-2 MF in the context of uncertainty representation.

In order to investigate also the relation among Type-2 MF and RFV let us refer to [13]. Let us consider newly a Type-2 fuzzy number by its $\alpha-$cuts

$$\left[[a_1^{(\alpha)}, a_2^{(\alpha)}], [a_3^{(\alpha)}, a_4^{(\alpha)}]\right].$$

Now, let us assign to each segment $[a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $[a_3^{(\alpha)}, a_4^{(\alpha)}]$ a pdf $f_L(\alpha, x)$ and $f_R(\alpha, x)$ respectively. Therefore, in the interval of confidence, the lower and the upper bounds become random variables. Figure 3 shows this concept, with F_L and F_R the probability distribution functions associated.

In [13] the authors also show that the envelope of a RFV is a Type-2 fuzzy number, and that, while the operations on the RFV are necessarily performed by sum-product convolution (since the confidence interval is represented by two pdfs), the operations on Type-2 fuzzy numbers can be performed by max-min convolution that corresponds to the use of the so called Extension Principle (EP) by Zadeh. They also show that the application of EP can be turned into working directly on $\alpha-$cuts, under the assumption of independent variables. Anyway, the use of RFV, is necessary when systematic errors or their corrections are partially unknown, so that standard approaches produce a wrong evaluation of uncertainty. In this case a particular class of rectangular Type-1 MF are embedded into the RFV, in order to model systematic errors or their uncomplete correction.

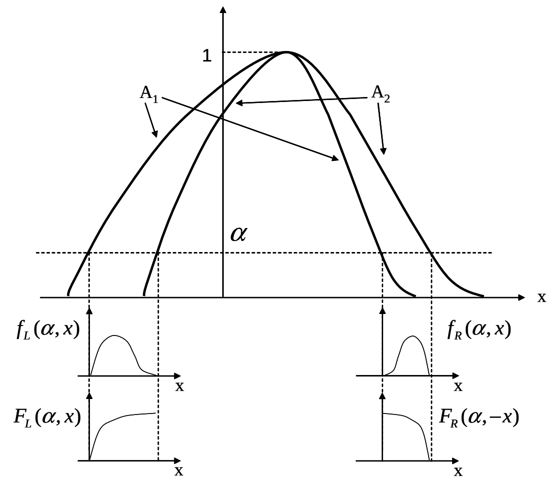


Fig. 3. Random Fuzzy Variables embedded in Type-2 MFs

In the following, we will assume that systematic errors are completely corrected, so that only random effects should be considered. So, in order to express uncertainty we consider the use of Interval Type-2 MF.

A. *Type-2 fuzzy numbers for type-B uncertainty handling*

Let us suppose that the measurement of the variable X is provided in the form $(X_0 \pm U_X)\,M$, where U_X is the expanded uncertainty of X, taken from manuals, calibration reports etc., and M is the measure unit (in the following omitted for notation simplicity). Then, in the case of incomplete knowledge of the pdf associated to X (i.e., in a type-B uncertainty expression), it is possible to build a gamma of pdfs starting from the declaration of X and by various assumptions and knowledge of the performed measurement process. Under these considerations, we can have the situation in Fig. 4.

Obviously, given the support $[x_1, x_2]$ the gaussian probability density function (gpdf) with $\sigma = (x_2 - x_1)/6$ (i.e., containing the 99.73% of the pdf in the support) is the most localized pdf around the central value $x^m = (x_1 + x_2)/2$, whereas the uniform probability density function (updf) with the same support is the least localized pdf.

Using the probability-possibility transformations introduced in [9] the Possibility Distributions (PDs) shown in Fig. 5 are obtained.

Note that there is a set of possible PDs associated with the support $[x_1, x_2]$ ranging from the PD related to a gpdf (that we will denote as Error function Possibility Distribution (EPD)) to the PD related to the updf (Triangular Symmetric possibility Distribution (TSPD) in the following), through the internal Parabolic Possibility Distribution (PPD). If one considers as the lower bound of this set of PDs the dotted one

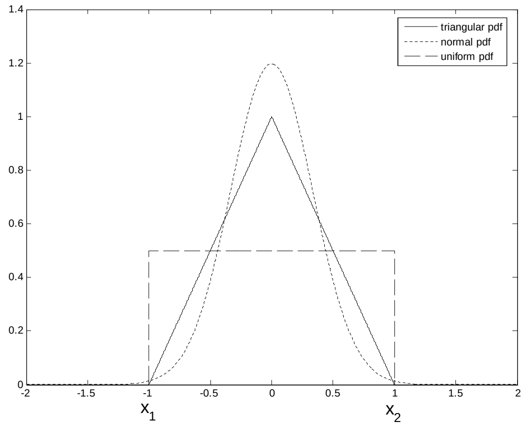


Fig. 4. Various pdfs associated to a type-B uncertainty evaluation

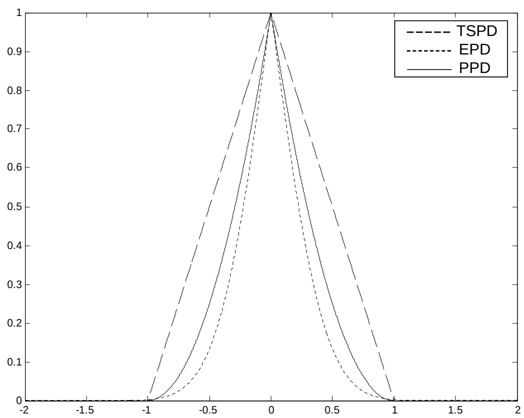


Fig. 5. PDs related to various pdfs associated to a type-B expression of uncertainty

(i.e., the EPD) and as the upper bound the dashed one (i.e., the TSPD) then a Type-2 fuzzy number is obtained, denoted by its Type-2 $\alpha-$cut, $\mathcal{A}_\alpha$,

$$\mathcal{A}_\alpha = [[x_1^{(\alpha)}, x_2^{(\alpha)}], [x_3^{(\alpha)}, x_4^{(\alpha)}]],$$

where $x_1^{(\alpha)}$, $x_2^{(\alpha)}$, $x_3^{(\alpha)}$, and $x_4^{(\alpha)}$ can be obtained [2] as

$$\begin{aligned} x_1^\alpha &= x_m - (1-\alpha)(x_m - x_1), \\ x_2^\alpha &= x_m - \frac{\sqrt{2}(x_2 - x_1)}{6} \cdot \text{erf}^{-1}(1-\alpha), \\ x_3^\alpha &= x_m + \frac{\sqrt{2}(x_2 - x_1)}{6} \cdot \text{erf}^{-1}(1-\alpha), \\ x_4^\alpha &= x_m + (1-\alpha)(x_2 - x_m), \end{aligned}$$

Now, each variable can be represented by means of this Type-2 MF, whose support is taken from the type-B expression of the measurement value. Otherwise, when a set of repeated observations for a given variable is available, then the IEC-ISO Guide recommends to perform a statistical analysis (i.e., a type-A uncertainty evaluation), so that a reliable estimation of the correct pdf can be extracted. Let us suppose, for example, that a gpdf is the best MLE for a given variable. Then, the use of the probability-possibility transformation involving only the gpdf, thus producing a EPD, leads to the degenerate Type-2 $\alpha-$cut

$$x_1^\alpha = x_2^\alpha = x_m - \frac{\sqrt{2}(x_2 - x_1)}{6} \cdot \text{erf}^{-1}(1-\alpha)$$

and

$$x_4^\alpha = x_3^\alpha = x_m + \frac{\sqrt{2}(x_2 - x_1)}{6} \cdot \text{erf}^{-1}(1-\alpha).$$

Note that, in this case, the Type-2 MF reduces to a Type-1 MF, embedding the information added by the statistical analysis. So, in order to propagate, through a function f, the uncertainty of each variable, the unique representation by means of Type-2 MFs can be adopted, so that the operations involved in f can be applied directly on the Type-2 $\alpha-$cuts $[[x_1^\alpha, x_2^\alpha], [x_3^\alpha, x_4^\alpha]]$ working as summarized in [13].

III. RESULTS

In this section we will provide an example of this kind of representation in the case of a simple tunable circuit. Let us consider Fig. 6. The two resistors are provided by the manufacturer in the form $R_i = (R_{i,0} \pm U_{R_i})\ \Omega$, with $R_{1,0} = 1\ \text{k}\Omega$ and $U_{R_1} = 10\ \Omega$ while R_2 is a variable resistor with a nominal value ranging over $[316\ \Omega, 10\ \text{k}\Omega]$ and a tolerance in the range $[\sim\ 16\ \Omega, 500\ \Omega]$. We assume a updf for R_1 and a gpdf for V_1 (provided by the manufacturer), while we cannot do any assumption on the correct pdf for the variable resistor R_2, so that a representation by means of a Type-2 MF is needed. In particular, the variability of R_2 generates a set of Type-2 MFs according to the nominal value and with different spread. So we can construct a Type-2 MF (degenerate or not) for each variable, obtaining Fig. 7 (a)-(b) and Fig. 8, where we have chosen only 10 different nominal values for R_2 for simplicity.

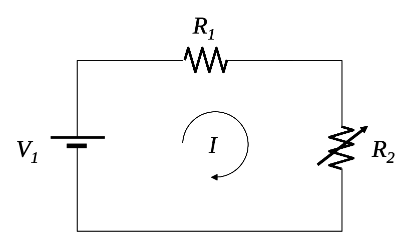


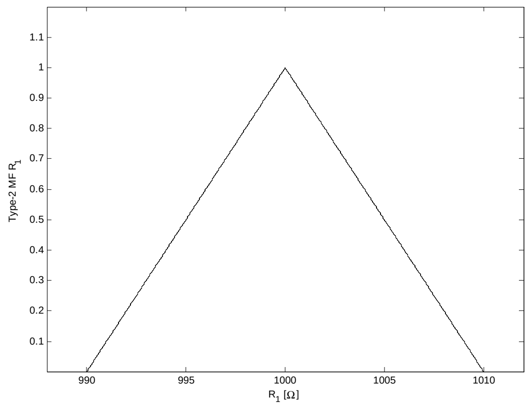
Fig. 6. An example of tunable circuit

Now, consider the I-V equation of the circuit

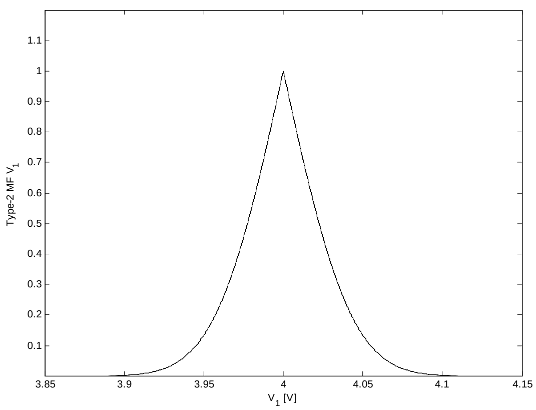
$$V_1 = I \cdot (R_1 + R_2)$$

and by inversion we get

$$I = V_1/(R_1 + R_2).$$



(a)



(b)

Fig. 7. Type-2 representations for (a) R_1 and (b) V_1

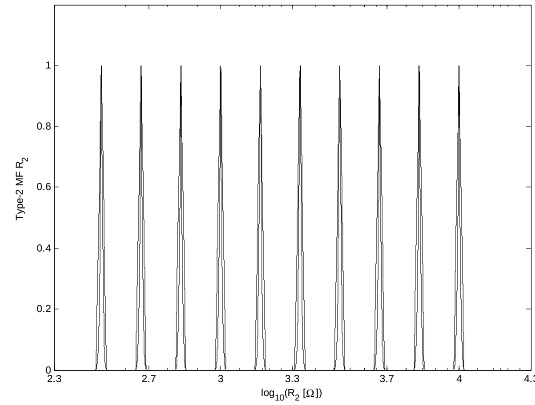


Fig. 8. Type-2 MFs for the variable resistor R_2 (x-logscale)

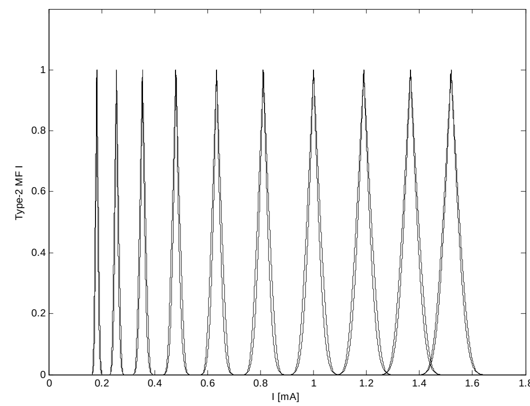


Fig. 9. Parameterized Type-2 MFs for the current I

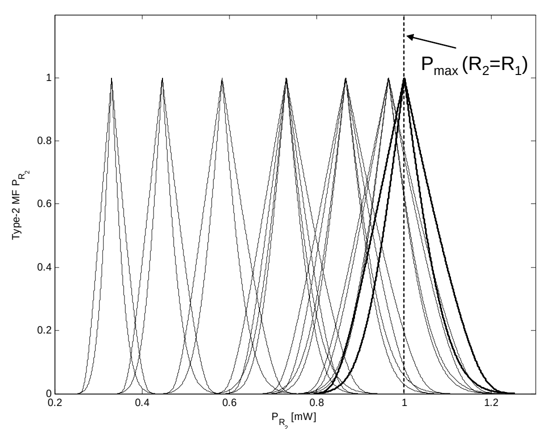


Fig. 10. Parameterized Type-2 MFs for the power dissipated on R_2, P_{R_2}

Owing to the variability of the nominal value of R_2 in the range $[\sim 316 \div 10000]\ \Omega$, various current values will be obtained.

First of all, let us compute a propagation through Type-2 MFs, so obtaining 10 different Type-2 MFs for the current $I(k)$, with k ranging over $[1, 10]$, owing to the variable resistor R_2, as shown in Fig. 9. Finally, let us evaluate the power dissipated on R_2, namely $P_{R_2}(k) = I^2(k) \cdot R_2(k)$ with k as above, thus obtaining Fig. 10.

As well known, the maximum nominal power P_{R_2} is reached when $R_2 = R_1$, as highlighted in Fig. 10 with the dotted vertical line. Anyway, the uncertainty sources of V_1 and of the two resistors strongly propagate through the current I and the power P_{R_2} so that the power can be greater than the maximum nominal one, or equivalently the maximum nominal power can be assigned to different nominal values of R_2 and R_1.

Let us now evaluate the combined standard uncertainty of P_{R_2} following IEC-ISO Guide recommendations. Firstly, we must evaluate the standard uncertainty of V_1, R_1, and R_2. Under the assumed pdf, by choosing a confidence level of $95\ \%$ for both the parameters V_1 and R_1 thus producing the

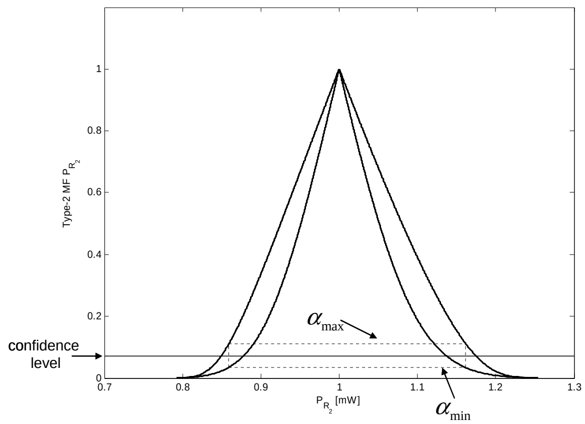


Fig. 11. Evaluation of the confidence level associated to the measurement of P_{R_2}

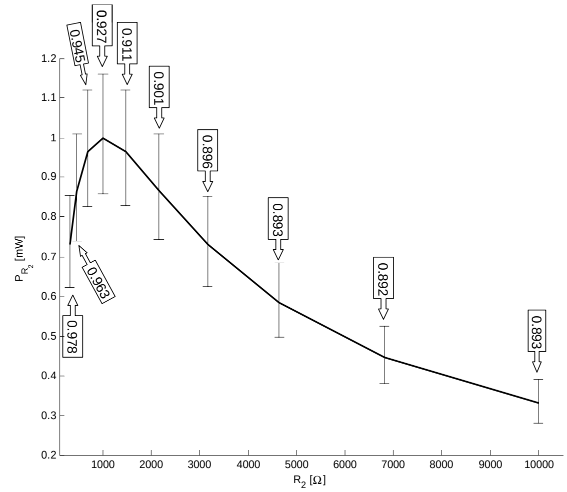


Fig. 12. Final representation of the power P_{R_2}, vs. the value of the resistor R_2, with the expanded uncertainty and the associated level of confidence

coverage factors $k_p = 1.64$ for R_1 and $k_p = 1.96$ for V_1, and, under the assumption of a unique coverage factor for R_2 equal to $k_p = 1.8$, we get

$$u_{R_1} = 10\ \Omega/1.64 \simeq 6.1\ \Omega,$$
$$u_{V_1} = 0.1\ \mathrm{V}/1.96 \simeq 0.05\ \mathrm{V},$$
$$u_{R_2} \in [16\ \Omega/1.8 \div 500\ \Omega/1.8] = [8.8\ \Omega \div 277.8\ \Omega]$$

So, under the assumption of uncorrelated variables, and if we denote as P_2 the power P_{R_2} (omitting measure unit for simplicity), we get

$$u_{P_2} = \sqrt{\left(\frac{\partial P_2}{\partial R_1}\right)^2 (u_{R_1})^2 + \left(\frac{\partial P_2}{\partial V_1}\right)^2 (u_{V_1})^2 + \left(\frac{\partial P_2}{\partial R_2}\right)^2 (u_{R_2})^2}$$

and, by inserting numerical values, we get

$$u_{P_2} \in [0.039,\ 0.045,\ 0.049,\ 0.050,\ 0.049,\ 0.044,\ 0.038,\ 0.031,\ 0.024,\ 0.018]\ \mathrm{mW}.$$

Now, choosing a coverage factor $k_p = 3$ for the variable P_{R_2} (see IEC-ISO Guide recommendations), we get the expanded uncertainty U_{P_2} as

$$U_{P_2} \in [0.116,\ 0.134,\ 0.147,\ 0.151,\ 0.146,\ 0.133,\ 0.114,\ 0.093,\ 0.072,\ 0.054]\ \mathrm{mW}.$$

Now, by comparing this set of values with the Type-2 MFs associated to P_{R_2} we can assign to each power the corresponding confidence level. This procedure is depicted in Fig. 11 only for the Type-2 MF related to the maximum nominal value. Anyway, this procedure has to be applied to all Type-2 MFs evaluated (ten in this example).

Note that, owing to the dispersion of the possible pdfs assigned to P_{R_2}, there is a range of possible confidence levels associated to a single measurement. Anyway, since the Type-2 MF is an Interval Type-2 fuzzy number, then we can assume as equally possible all the confidence levels and compute the mean value as the most representative. By performing this operation we get the result of Fig. 12.

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