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On a generalized $T-$norm for the representation of uncertainty propagation in statistically correlated measurements by means of fuzzy variables

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Abstract – *The problem of uncertainty representation and propagation in the context of statistically correlated variables is commonly addressed by means of Montecarlo simulation as recommended in IEC-ISO Guide. Moreover, in a recent literature, fuzzy sets have proved to be a valid alternative in the case of independent variables. Unfortunately, the problem of modelling statistically correlated variables, by means of fuzzy sets, is still an open problem. Since it is well known that $T-$norms are the natural way of combining fuzzy variables into a nonfuzzy function f, in this paper, we investigate how to generalize the class of $T-$norms, making it dependent from correlation coefficient in order to emulate different statistical correlation degree among variables. In order to validate the model a comparison with Central Limit Theorem will be accomplished in the case of zero correlation while a practical example will be provided in order to compare the proposed method with Montecarlo simulation and with that obtained by uncertainty propagation described in IEC-ISO Guide.*

Keywords – *Uncertainty representation, $T-$norm, fuzzy variables.*

I. INTRODUCTION

In this paper we address the problem of uncertainty representation and propagation in a context of statistically correlated variables. This problem has been widely investigated in literature in the context of probability theory, by means of joint probability density function, and implemented by Montecarlo simulation as recommended in [1]. Surely, Montecarlo simulation, as noted in [1], allows to implement the so called *probability distribution propagation*, instead of *uncertainty propagation*, avoiding Taylor approximation [2] of the output function f. Moreover, this approach allows to simultaneously work with all confidence intervals (so with all confidence levels) instead with a single one. In order to emulate this characteristic, but avoiding the time-consuming calculi involved in a Montecarlo simulation, in a recent literature, many alternative methods have been investigated and applied in the context of uncertainty modelling, such as Random Fuzzy Variables (RFV) [3–5], Fuzzy approaches [6–8], Type-2 fuzzy variables [9], etc. Really, fuzzy theory allows to model and propagate uncertainty for all confidence intervals simultaneously, avoiding the handling of a very high number of samples needed for a correct representation of the output variable.

Anyway, at this moment, the problem of modelling statistically correlated variables by means of fuzzy sets, in a formula that directly depends on the correlation coefficient ρ is still an open problem. Probably, the way in which probability theory is mapped into possibility theory (i.e., fuzzy theory) is crucial in order to build a generalized representation of uncertainty propagation. In particular, in this paper, we refer totally to the probability-possibility transformations introduced in [10] and already used by the authors in [9].

Fuzzy sets are in many practical cases one of the best methods for uncertainty representation (from both computational aspects and chance of a direct evaluation), so that we are interested in how fuzzy sets can be aggregated in order to evaluate a function f of fuzzy variables originated from statistically correlated random variables; this means that $T-$norms, representing the way fuzzy sets are aggregated, have to be investigated.

What we expect, according to the $T-$norm used, is that the final result could emulate or not the result obtained by a Montecarlo simulation. It is well know that the Extension Principle (EP) by L. Zadeh [11], is the natural way of combining fuzzy variables into a nonfuzzy function f. Also, it is well known that implementation of EP is quite complex. So, recently, the equivalence between applying EP and operating directly on $\alpha-$cuts have been proved (Nguyen's Theorem [12]), when $T-$norm is implemented by the *minimum* operator ($T_{\min}$ in the following). In this way, calculus is strongly simplified, but this result does not hold for different types of $T-$norms. However, another result (Fuller's Theorem [13, 14]) generalizes that equivalence to the case of a general $T-$norm.

The crucial point is that by using different kind of $T-$norms we can emulate different statistical correlation degree among variables. Obviously, the equivalence is not exact, for all correlation coefficients ρ, but the error is always very low (or zero in some cases). Moreover, building a particular ρ-dependent $T-$norm (T_ρ in the following) we will show that working on α-cuts is also ρ-dependent (as expected) and a direct formula will be derived. In this way, we will provide an uncertainty propagation model, by means of fuzzy variables, able to embed correlation degree as a free parameter of the model.

So, in Section II, we will recall briefly EP, concepts of $T-$norm and $T-$conorm, Nguyen's Theorem and Fuller's Theorem. In Section III, we will examine the effect of statistical correlation among variables on the EP, when a certain class of probability-possibility transformations [10] are used. In Section IV, we will describe the generalized T_ρ proving that it is a $T-$norm. Finally, in Section V we will validate the approach in the case of independent RVs by a comparison with results obtained by Central Limit Theorem, while in Section VI, we will perform a comparison with Montecarlo simulation, performed following recommendations in [1], in the context of statistically correlated measurements.

II. PRELIMINARIES

The concepts of fuzzy sets have been introduced by Zadeh [15, 16] as an extension of the traditional concept of membership of a variable a to a set A. In crisp set theory this membership is represented by a one ($a \in A$) or by a zero ($a \notin A$), whereas in fuzzy set theory it can be modelled by a Membership Function MF $\mu_A(a)$ such that $0 \leqslant \mu_A(a) \leqslant 1$, with $\mu_A(a)$ convex and normal (there exists at least one value b such that $\mu_A(b) = 1$. The $\alpha-$level set (or $\alpha-$cut) of A is a nonfuzzy set, denoted by A_α, defined as $A_\alpha = \{a|\mu_A(a) \geqslant \alpha\}$. The MF is the natural way to extend the concept of confidence interval and confidence level, as recommended in [2], so that in the following we will examine in detail the way fuzzy numbers can be combined in a nonfuzzy way (i.e., through a function f). The standard way is formalized by the EP introduced by Zadeh in [11]. We will recall EP in the following.
Let f be a mapping from $X_1 \times \cdots \times X_r$ to a set Y such that $y = f(x_1, \ldots, x_r)$, for $y \in Y$ and $x_i \in X_i$, $i = 1, \ldots, r$. Then the EP allows us to build the MF of y on B, $\mu_B(y)$, starting from the MFs of x_i on A_i, $\mu_{A_i}(x_i)$, through f such that

$$\mu_B(y) = \sup_{\substack{x_1,\ldots,x_r \\ y=f(x_1,\ldots,x_r)}} T\text{−norm}\left(\mu_{A_1}(x_1), \ldots, \mu_{A_r}(x_r)\right) \qquad (1)$$

and $\mu_B(y) = 0$ if $f^{-1}(y) = \emptyset$. Since the application of EP is quite complex, the compatibility of the EP with the $\alpha-$cuts representation has been investigated. This equivalence is also known as Nguyen's Theorem [12] which states that under the assumption that $T-$norm $\equiv T_{\text{min}}$ the following relation holds

$$[f(A_1, \ldots, A_r)]_\alpha = f\left(A_{1\alpha}, \ldots, A_{r\alpha}\right).$$

This means that the operations involved in f can be applied directly on the $\alpha-$cuts. The proof of this result can be found in [12]. Now, we need a generalization of Nguyen's Theorem in the case of a general $T-$norm. First of all, recall that

Definition II.1: A $T-$norm is a function $T : [0,1] \times [0,1] \to [0,1]$ with the following properties.

(t1) Commutativity: $T(a, b) = T(b, a)$.
(t2) Monotonicity: $T(a, b) = T(c, d)$ if $a = c$, $b = d$.
(t3) Associativity: $T(a, T(b, c)) = T(T(a, b), c)$.
(t4) Null element: $T(a, 0) = 0$.
(t5) Identity element: $T(a, 1) = a$.

The dual operation, called $T-$*conorm*, satisfies the following relations.

(s1) Commutativity: $S(a, b) = S(b, a)$.
(s2) Monotonicity: $S(a, b) = S(c, d)$ if $a = c$, $b = d$.
(s3) Associativity: $S(a, S(b, c)) = S(S(a, b), c)$.
(s4) Null element: $S(a, 1) = 1$.
(s5) Identity element: $S(a, 0) = a$.

Among various kind of $T-$norms in the following we address the $T_{\text{min}} = \min\{a, b\}$ and $T_{\text{prod}} = a \cdot b$ since they have a direct probabilistic counterpart. Then, we will consider only T_{max} conorms since all the results we use are valid only for this operator.

Now, in order to generalize the Nguyen's Theorem so that we can still establish, for a wider class of variables, an equivalence between the EP and operating on $\alpha-$cuts, we recall the Fuller's Theorem [14]. It states

Theorem II.1 (Fuller's Theorem) Let $X \neq \emptyset$, $Y \neq \emptyset$ be sets and let T be a $T-$norm. If $f : X \times Y \to Z$ is a two-place function and A and B are fuzzy sets on X and Y respectively, then a necessary and sufficient condition for the equality

$$[f(A, B)]_\alpha = \bigcup_{T(\xi,\eta)\geqslant\alpha} f\left(A_\xi, B_\eta\right), \quad \alpha \in (0, 1],$$

is that, for each $z \in Z$, $\sup_{f(x,y)=z} T\left(A(x), B(y)\right)$ is attained. Despite the apparent complexity of this result, its implementation is quite simple and efficient.
In [14] the authors show that if the $T-$norm is T_{min} then the classical result by Nguyen follows.

In the following, we will apply Fuller's Theorem in order to work with a ρ-dependent $T-$norm and consequently with statistically correlated variables, but still operating on $\alpha-$cuts. To do so, firstly we will investigate the equivalence between a certain $T-$norm in fuzzy sets theory and analogous operators, in probability theory, in the case of correlated random variables.

III. EFFECT OF STATISTICAL CORRELATION ON EP

Let us consider only symmetric pdfs in the following. Using probability-possibility transformations in [10] and denoting with x^* the central value (corresponding to the modal value $\bar{x}$) of the pdf with compact support $[x_1, x_2]$ the following relations hold for the Possibility Distribution PD $\pi(x)$ of x (i.e., the MF of x)

$$\begin{aligned} \pi_-(x) &= 2F(x), \quad x_1 \leqslant x \leqslant x^*, \\ \pi_+(x) &= \pi_-(x), \quad x^* \leqslant x \leqslant x_2 \end{aligned} \qquad (2)$$

where π_+ and π_- denote the left and the right part of π with respect to the central value x^* and $F(x)$ is the cumulative probability distribution of the Random Variable RV x. Note that there is a direct relation between the PD and $F(\cdot)$. Using (2), the effect of statistical correlation on the shape of $\pi(x)$ can be easily investigated. Firstly, we have to recall the concept of joint pdf and joint PD. Let us consider two statistically correlated RVs, x and y with correlation coefficient ρ. The joint probability that x and y are below some x_0 and y_0, that corresponds to $P(x \leqslant x_0, y \leqslant y_0) = F(x_0, y_0)$ is called *joint cumulative probability distribution*. Analogously, Zadeh in [11] introduced the concept of *joint possibility distribution* $\pi(x, y)$. We recall it in the case of two variables.

Let x and y be two variables with universes of discourse U_x and U_y, then the variable $z = f(x, y)$ will be referred as a *joint variable*. The universe of discourse of z is the cartesian product $U_z = U_x \times U_y$. The MF μ_z is defined as $\mu_z : U_x \times U_y \to [0, 1]$ and is given by (1). The fuzzy set $\pi(x, y) = T_{\text{norm}}(\pi(x), \pi(y))$ is called *joint possibility distribution*. Obviously, since equation (2) holds, we also get

$$\pi(x, y) = 2 \int_{y_1}^{y} \cdot\, 2 \int_{x_1}^{x} f(u, v) du\, dv = 4 \cdot F(x, y),$$

with $x_1 \leqslant x \leqslant x^*$ and $y_1 \leqslant y \leqslant y^*$ and easily extended by symmetry in the whole domain.

Let us consider now the two cases $\rho = 0$ and $\rho = 1$ separately.

a) Independent RVs $\rho = 0$. In this case it is well known (see e.g., [17]) that $F(x,y) = F(x) \cdot F(y)$ so that $\pi(x, y) = 2F(x) \cdot 2F(y) = \pi(x) \cdot \pi(y)$. Consider now the EP in the general form. Given a function $z = f(x, y)$, we get $\pi_z(z) = \bigvee_{(x,y)\ z=f(x,y)} \pi_x(x) \bigwedge \pi_y(y)$, where $\bigvee$ denotes the $T-$conorm and $\bigwedge$ denotes the $T-$norm. So, it is natural to consider $T_{\text{prod}}(x, y)$ as $T-$norm in this case, getting $\pi_z(z) = \bigvee_{(x,y)\ z=f(x,y)} \pi_x(x) \cdot \pi_y(y)$ If $T_{textmax}$ is used as $T-$conorm (generally sup instead of max for PD with non compact support) then we get the sup-convolution operator, very similar to the convolution in the probability theory, and we finally get

$$\pi_z(z) = \sup_{(x,y)\ z=f(x,y)} \pi_x(x) \cdot \pi_y(y).$$

b) Linear correlated RVs $|\rho| = 1$. In this case, using that $y = ax + b$ we get $P(x \leqslant x_0, y \leqslant y_0) = P(x \leqslant x_0, y \leqslant ax_0 + b) = F(x_0, ax_0 + b)$. This result suggests that the joint pdf is $f(x)$ or $f(y)$ according to the value of a and b. In particular, in the case of a sum of totally correlated RVs x and y ($z = x + y$), we get that $P(z \leqslant z_0) = F_z(z_0) = P((x + a \cdot x + b \leqslant z_0) = P((a + 1) \cdot x + b \leqslant z_0) = P\left(x \leqslant \frac{z_0 - b}{a+1}\right) = F_x\left(\frac{z_0-b}{a+1}\right)$. So, we can derive the pdf of the RV $z = x + y$ by the pdf of x or y. Note that, in the case of x, $y \sim U([0, 1])$ (i.e., with uniform pdf) then $z \sim U([b, a + b + 1])$. So, in this case the use of $T_{\min}$ as $T-$norm seems to be the natural choice, since the *minimum* operator leaves unchanged the shape of the fuzzy sets belonging to the same class.

Now, arguing as above for the $T-$conorm we get

$$\pi_z(z) = \sup_{(x,y)\ z=f(x,y)} \min\left(\pi_x(x), \pi_y(y)\right).$$

In the following section, we will generalize this approach in the general case of a correlation coefficient $\rho \in [0, 1]$.

IV. A GENERALIZED $T-$NORM

Let us consider now the $T-$norm T_ρ defined by

$$T_\rho(a, b) = (1 - \rho)\, T_{\text{prod}}(a, b) + \rho\, T_{\min}(a, b), \quad \rho \in [0, 1]$$

which is a convex combination of the two classical $T-$norms T_{prod} and $T_{\min}$. First of all, we have to verify that $T_\rho(a, b)$ is a $T-$norm. Let us prove that $T_\rho(a, b)$ satisfies (t1) – (t5).

- (t1) By symmetry of $T_{\text{prod}}(a, b)$ and $T_{\min}(a, b)$ (t1) follows immediately.
- (t2) By monotonicity of both T_{prod} and $T_{\min}$ (t2) follows.
- (t3) From associativity of $T_{\text{prod}}(a, b)$ and $T_{\min}(a, b)$ (t3) follows immediately.
- (t4) Since $T_{\text{prod}}(0, b) = T_{\text{prod}}(a, 0) = 0$ and $T_{\min}(0, b) = T_{\min}(a, 0) = 0$ (t4) follows.
- (t5) Since $T_{\text{prod}}(1, b) = b$ and $T_{\text{prod}}(a, 1) = a$ and $T_{\min}(1, b) = b$ and $T_{\min}(a, 1) = a$ we get (t5).

Now, we can apply Theorem II.1 with T_ρ. So, we have to solve $T_\rho(\xi, \eta) \geqslant \alpha$ and insert it into

$$[f(A, B)]_\alpha = \bigcup_{T(\xi,\eta)\geqslant\alpha} f\left(A_\xi, B_\eta\right), \quad \alpha \in (0, 1].$$

Now, we have to consider separately the cases $\xi \leqslant \eta$ and $\xi > \eta$. In the first case, we get $(1 - \rho)\, \eta\, \xi + \rho\, \xi \geqslant \alpha$ and so, we obtain

$$\xi \geqslant \min\left(\max\left(\frac{\alpha}{(1-\rho)\,\eta + \rho}, 0\right), 1\right), \quad \eta \in [\alpha, 1].$$

Otherwise, if $\xi > \eta$, we get

$$\eta \geqslant \min\left(\max\left(\frac{\alpha}{(1-\rho)\xi + \rho}, 0\right), 1\right), \quad \xi \in [\alpha, 1].$$

So, we get

$$[f(A, B)]_\alpha = \bigcup_{\substack{\xi\leqslant\eta \\ \xi\geqslant\min\left(\max\left(\frac{\alpha}{(1-\rho)\,\eta+\rho}, \alpha\right), 1\right) \\ \xi>\eta \\ \eta\geqslant\min\left(\max\left(\frac{\alpha}{(1-\rho)\,\xi+\rho}, \alpha\right), 1\right)}} f\left(A_\xi, B_\eta\right), \tag{3}$$

for $\alpha \in (0, 1]$. Note that, when $\rho \to 0$ then $T_\rho(\xi, \eta) \to T_{\text{prod}}(\xi, \eta)$, in fact we get

$$[f(A, B)]_\alpha = \bigcup_{\xi\in[\alpha,1]} f\left(A_\xi, B_{\alpha/\xi}\right), \quad \alpha \in (0, 1].$$

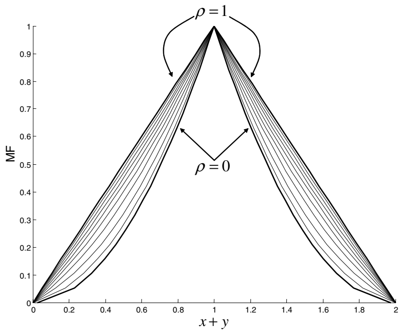


Fig. 1. MF of a sum of correlated RVs vs. their correlation coefficient ρ.

Otherwise, if $\rho \to 1$ then $T_\rho(\xi,\eta) \to T_{\min}(\xi,\eta)$, in fact we get

$$[f(A,B)]_\alpha = \bigcup_{\xi\in[\alpha,1],\ \eta\in[\alpha,1]} f\left(A_\xi, B_\eta\right), \quad \alpha \in (0,1],$$

corresponding to classical formulation of Nguyen's Theorem. Note that equation (3) provides a closed formula to compute directly α-cuts of the output variable without applying EP on the input variables.

By using above results, we generalize approach described in Section III. If we use $T_\rho(a,b)$ instead of any other $T(a,b)$, we get

$$\pi_z(z) = \sup_{(x,y)\ z=f(x,y)} T_\rho\left(\pi_x(x), \pi_y(y)\right).$$

Let us apply this approach to the sum of two uniform RVs $x, y \sim U([0,1])$, varying the correlation coefficient ρ in $[0,1]$. As expected, we get two Triangular PDs (TPD) π_x and π_y with the same support $[0,1]$. The PD of the variable $z = x + y$ is shown in Fig. 1, for some different values of ρ between 0 (corresponding to T_{prod}) and 1 (corresponding to $T_{\min}$). Note that, in the case of uncorrelation ($\rho = 0$), we get a non triangular MF for z while in the case of total correlation ($\rho = 1$), we get a TPD. The cases of $\rho \in (0,1)$ are between the two external cases $\rho = 0$ and $\rho = 1$. This means that, as a counterpart of the Central Limit Theorem (CLT), if we sum two independent RVs with TPD, by using Dubois probability-possibility transformations, we do not obtain a TPD. From these aspects, the class of TPDs is not closed under T_ρ, whereas it is closed under classical $T_{\min}$. Really, we do not need a class of PDs closed under a certain $T-$norm, which would be obviously desirable from computational aspects, but we need a $T-$norm that emulates the behavior of a probabilistic approach. So, if we want to perform operations on fuzzy variables, that are as much similar as the ones performed on RVs, then the T_ρ is needed. In the following section we will compare the proposed uncertainty model with the application of CLT on the average of N independent and identically-distributed random variables.

V. VALIDATION BY A COMPARISON WITH CLT

Let us consider N independent and identically-distributed random variables (i.i.d.) $x_i \sim U([0,2])$, $i = 1,\ldots,N$. Consider now the variable $y = \frac{\sum_{i=1}^N x_i}{N}$. It is well know that the pdf of y, $f_Y(y)$, tends to be normally distributed with unitary mean $\eta_y = 1$ and variance σ_y^2 given by $\sigma_y^2 = \frac{1}{N^2}\sum_{i=1}^N \sigma_{x_i}^2$. Now, by applying $T_\rho-$norm with $\rho = 0$ to the variable x_i, for $N = 2,\ 6,\ 10$ we compare the pdf obtained by applying CLT with the PD obtained by the proposed model. The results are shown in Fig. 2(a)-(b).

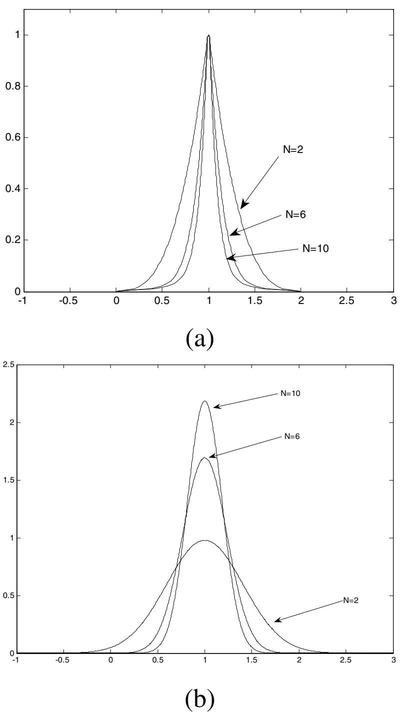


Fig. 2. (a) PDs obtained by $T_\rho-$norm and (b) pdfs obtained by CLT.

Note that there is a great agreement in this case, proving that T_{prod} is the best counterpart of a probabilistic approach. In the following section we provide some simulation results of the proposed uncertainty model and a comparison with Montecarlo simulation in the case of statistically correlated measurements.

VI. SIMULATION RESULTS

Let us consider the case of a series of two resistors, R_1 and R_2 after a measurement performed by a comparison with a known reference standard R_s, with nominal value $R_{s\,0}$ and standard uncertainty u_{R_s}. We know [2] that under the assumption that the uncertainty of the comparison is much smaller than the uncertainty of the reference standard, then the correlation coefficient $\rho(R_1,R_2) \simeq 1$. Really, the expression of $\rho(R_1,R_2)$ is given by

$$\rho(R_1,R_2) = \left[1 + \left(\frac{u_\alpha}{u_{R_s}/R_s}\right)^2\right]^{-1} \tag{4}$$

where $R_i = \alpha_i R_s$, $i = 1, 2$. Note that if the uncertainty $u_\alpha << u_{R_s}/R_s$, then we get $\rho \simeq 1$. In this example, we will assume that $u_{\alpha_i} = u_\alpha$ does not depend on the factor α, and it is the same for each resistor. For example, we will assume that $u_\alpha = 10^{-2}$. So, if we suppose also that $u_{R_s}/R_s = 10^{-2}$, we get $\left(10^{-2}/10^{-2}\right)^2 = 1$ and consequently $\rho(R_1, R_2) \sim 0.5$. Obviously, the expression of u_α strictly depends on the procedure implemented for computing the value of α. This is only an illustrative example.

If we have two different values of α, namely $\alpha_1 = 2^{-1}$ and $\alpha_2 = 2^{-4}$, then we have $R_1 = 2^{-1} \cdot R_s$ and $R_2 = 2^{-4} \cdot R_s$. Note that, since $R_i = \alpha_i \, R_s$, under the assumption of uncorrelation between α_i and R_s and following recommendations in [2], we can express the standard uncertainty of each R_i as $u_{R_i} = \sqrt{(R_{s\,0})^2 \cdot u_\alpha^2 + \alpha_{i\,0}^2 \cdot u_{R_s}^2} = 10^{-2} R_{s\,0} \sqrt{(1+\alpha_i^2)}$, where $\alpha_{i\,0}$, $i = 1, 2$ are the two nominal values for α_i, that are, in this case, 2^{-1} and 2^{-4} respectively. Then, we finally get $u_{R_i} = 10^{-2} \, R_s \, [1.118 \; 1.002]$. Now, suppose that the two resistors are linked using a parallel wiring scheme through conductors having a negligible resistance, so as to achieve a resistance R_p with nominal value given by $1/R_{p\,0} = \sum_{i=1}^{2} 1/R_{i\,0} = 1/R_{s\,0}\,(2+16) \simeq 18/R_{s\,0}$, that corresponds to $R_{p\,0} = R_{s\,0}/18$. The standard uncertainty of R_p, obtained following IEC-ISO Guide procedure, is equal to $0.89 \cdot 10^{-2} \cdot R_s$. At this point, even if we suppose $R_{s\,0} = 1\,\Omega$, we are not able to assign a confidence level to this uncertainty, because we do not know the pdf of R_p and we cannot assume it a gaussian pdf. If it was gaussian, then, with a coverage probability of 95%, we get a coverage interval $[0.0381, 0.0730]\ \Omega$. In the following, we will compare this interval with the uncertainty representation obtained by means of T_ρ (i.e., fuzzy sets) and by the propagation of distributions by a Montecarlo simulation.

A. The fuzzy approach

In order to perform uncertainty propagation by means of T_ρ and fuzzy sets, first of all, we have to design the MF for each input variable R_i, $i = 1, 2$. Since the standard uncertainty for each variable is known, and since we can suppose a uniform pdf for each R_i, $i = 1, 2$, then, with a coverage probability of 95% for R_1 and R_2, we get a coverage factor of $\sqrt{3} \cdot 0.95$ and we get the support for R_1 and R_2 shown in Fig. 3(a)-(b). Note that we get two TPDs, since we have suppose the two variables R_1 and R_2 as uniform RVs. Now we apply the T_ρ in order to build the MFs of $R_p = (R_1 \cdot R_2)/(R_1 + R_2)$, thus obtaining Fig. 3(c), with $\rho = 0.5$. We compute the α-cuts of the variables R_p by implementing equation (3).

Note that the statistical correlation among R_1 and R_2 produces a slight deviation of the resulting MF from a triangular one. The MF is computed for 100 different levels of α in $[0, 1]$.

Now, given a coverage probability of 95%, we have to evaluate the interval corresponding to $\alpha = 1 - 0.95 = 0.05$. Really, at this point, we can obtain all the confidence intervals associated to every coverage probability. In this case we obtain a confidence interval of $[0.0428,\ 0.0678]\ \Omega$.

B. The Montecarlo simulation

In order to perform the probability distribution propagation by means of a Montecarlo simulation, first of all, we have to build the joint pdf for R_1 and R_2, since the two variables are correlated with correlation coefficient $\rho = 0.5$. As noted in Supplement [1], we can always build a bivariate Gaussian, with given marginal pdfs and correlation coefficient. In Fig. 4, we show an example of simulated dependent random variables with uniform marginal pdfs and $\rho = 0.5$. Note the slight non

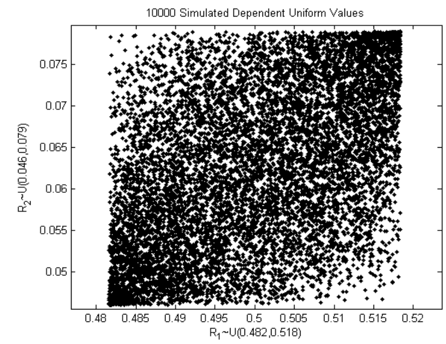


Fig. 4. Simulated dependent uniform random variables.

uniformity of the joint variable due to the correlation coefficient equal to 0.5. Now, we can evaluate the cumulative probability distribution for $R_p = (R_1 \cdot R_2)/(R_1 + R_2)$ obtained according to recommendations in [1]. We expect, from probability theory, that the pdf of R_p has a weak asymmetry so that, given a coverage probability and following [1], we should compute the shortest coverage interval for R_p from its cumulative probability distribution. However, if we consider level $\alpha_p = (1-p)/2$, we can directly compute the values $F^{-1}(\alpha_p)$ and $F^{-1}(p + \alpha_p)$, as lower and upper bound of the coverage interval, where F denotes the cumulative probability distribution of R_p. This situation is shown in Fig. 5 where the dotted lines determine the bounds of the coverage interval. Note that in this kind of approach α_p is approximately 0.025. In this case we have a confidence interval equal to $[0.0428,\ 0.0678]\ \Omega$ that

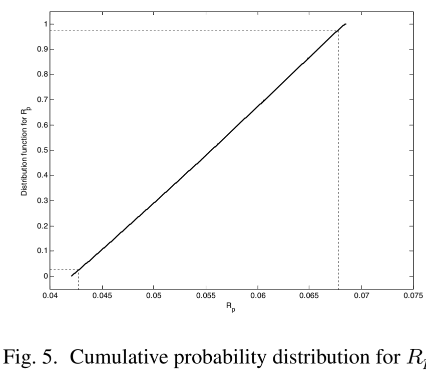
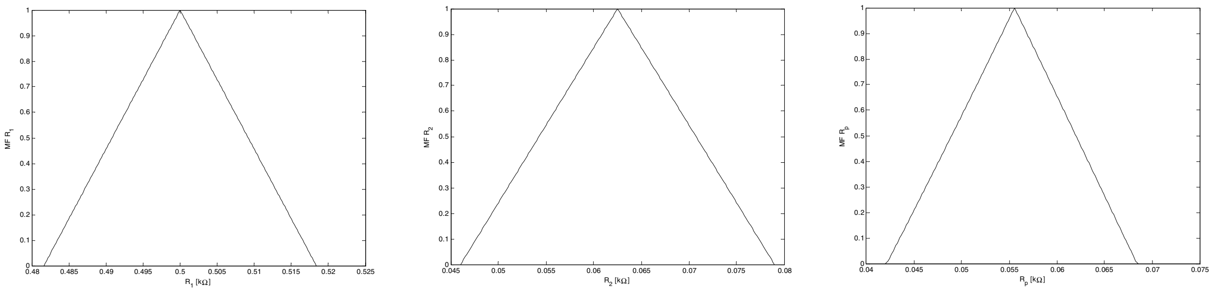


Fig. 5. Cumulative probability distribution for R_p.



(a) (b) (c)

Fig. 3. MFs for R_1, R_2, and R_p.

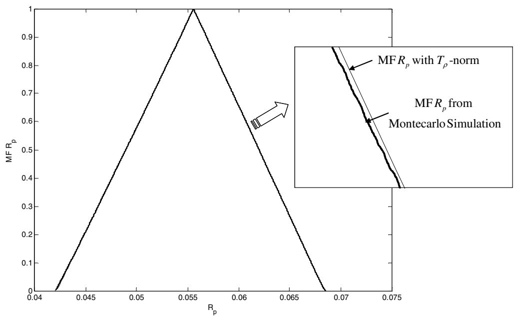


Fig. 6. Comparison between MF obtained by T_ρ and by Montecarlo simulation.

is the same of the fuzzy approach. Recall that the confidence interval estimated by IEC-ISO Guide, under the assumption of Gaussian pdf for R_p, evaluated in Section VI, was strongly overestimated.

Finally, we want also to compare result obtained by Montecarlo simulation with the representation by fuzzy numbers using T_ρ for all confidence level. To do this, we have to transform the cumulative probability distribution $F(R_p)$ into a MF. Owing to the weak asymmetry, we can simply consider as central value the mean value that corresponds to $R_p = 0.0556\ \Omega$. So, we obtain the comparison in Fig. 6 where the two MFs obtained from T_ρ and Montecarlo simulation respectively, are shown. Note the negligible difference between the two plots. Despite very similar results in the confidence intervals provided by the two methods, the simulation performed by means of T_ρ is very faster. Moreover, recall that, only if the number of samples tends to be very high, then Montecarlo result tends to match the true value.

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