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# A comparison between different methods for processing the random part of Random-Fuzzy Variables representing measurement results

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***Abstract –*** *In the recent years, fuzzy variables and random-fuzzy variables have been proposed to represent the measurement results with their associated uncertainty. However, up to now, the different authors do not yet agree in the mathematical way fuzzy variables should be composed together, so that different approaches have been proposed. This paper compares these approaches, in order to find their advantages and disadvantages and shows a new proposal, that is supposed to overcome, hopefully, the disadvantages of the original ones.*

***Keywords –*** *Uncertainty estimation; Random-Fuzzy Variables; t-norms; random contributions*

## I. INTRODUCTION

Traditionally, fuzzy variables are used in control systems. However, in the recent years, fuzzy variables have been also introduced in a very different context: metrology and measurements [1–7].

Since the '90s, when the GUM [8] has been issued, the mathematical theory considered in modeling measurement uncertainty has been the probability theory. Following the GUM and this theory, a measurement result is modeled as a random variable, mathematically represented by a probability density function. This representation, however, has some drawbacks.

First of all, the probably most important issue, is that probability theory has been developed to represent random phenomena. On the contrary, the phenomena affecting a measurement procedure and hence producing a specific effect on the measurement result, are not only of random nature, but also of systematic and, sometimes, unknown nature. It follows that, when also these last contributions are present, the probability theory is not longer adequate to model the measurement procedure.

Another drawback is that it is quite difficult to combine probability density functions and hence estimate how the different contributions to uncertainty propagate through the measurement process and affect the final result. For this reasons, either numerical methods are introduced, such as the Monte Carlo simulations, or simplifications are adopted, such as the mere propagation of the first two moments (mean and standard deviation) of the distributions. Both solutions, however, show some negative consequences: Monte Carlo simulations are generally time-consuming procedures; considering only the first two moments may lead to a wrong evaluation of the measurement uncertainty, mainly due to the application of the central limit theorem outside its validity conditions [1, 2].

These reasons have led researchers to look for different theories to handle measurement results and measurement uncertainty. In the last 10 years, the possibility theory and the theory of evidence have been investigated by different authors [1–7, 9–11].

Their common idea is that measurement results and their associated measurement uncertainty can be effectively represented by fuzzy variables; moreover, when measurement procedures are considered, the combination of fuzzy variables can be obtained by applying suitable fuzzy operators, in order to obtain the final measurement result directly in terms of a fuzzy variable. The advantage is immediately clear: thanks to fuzzy operators, fuzzy variables can be easily combined with each other.

However, the approaches proposed by the different authors, still disagree on two points:

- the kind of fuzzy variables to be used (fuzzy variables of type 1 or fuzzy variables of type 2);
- the choice and definition of the fuzzy operators to be applied to combine fuzzy variables.

As far as the first point is concerned, it can be noted that fuzzy variables of type 1 are generally considered to replace the random variables. In this respect, they still follow the hypothesis of the GUM [8] that only considers random phenomena, and assumes all the other ones negligible.

It is obvious that, under the hypothesis of only random contributions to uncertainty, the use of fuzzy variables of type 1 is sufficient. However, considering the different mathematical assumptions that are behind fuzzy and random variables, and the great flexibility shown by fuzzy variables, it seems quite reductive to consider fuzzy variables as a simple alternative to random variables for representing the same random phenomena.

On the other hand, fuzzy variables of type 2 allow one to represent different kinds of incomplete information, and they have been considered to model uncertainty, since not only random phenomena, but also unknown and systematic contributions do generally affect a measurement procedure [1]. In particular, among the fuzzy variables of type 2, the Random-Fuzzy Variables (RFVs) have been suitably defined in order to represent systematic and unknown contributions with their inner membership function, and random contributions with their outer membership function (Fig. 1) [1–3].

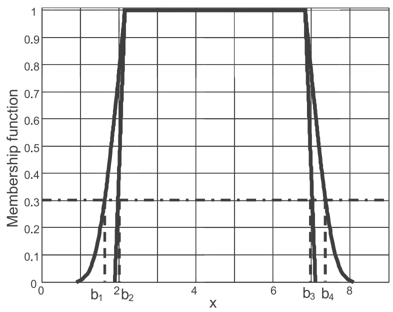


Fig. 1. Example of RFV

Therefore, it can be concluded that RFVs, that is fuzzy variables of type 2, must be used to represent measurement results and their associated uncertainty, thus closing the first of the aforementioned points.

On the contrary, the second point, that is the definition of the proper fuzzy operators, is still an open problem and will be discussed in this paper. Let us consider, however, that this problem is only related to the combination of the random part of the RFVs, since the mathematical approach followed in [1, 9, 12] to combine the non-random part of the RFVs is a generalization of the well known mathematics of the intervals and appears to be universally accepted. On the contrary different approaches have been proposed for the combination of fuzzy variables when they are employed to represent only random phenomena [1, 4, 6, 9–11]. Most of them are based on t-norms [13], while one is based on averaging operators [1, 9, 13].

In this paper, these methods are compared and discussed. Finally, a new proposal is presented, with the aim of finding a unified approach and mitigate the disadvantages of the original ones.

## II. EXPERIMENTAL COMPARISON AMONG THE EXISTING APPROACHES

In this section, the methods proposed by different authors for the composition of fuzzy variables are compared. Let us remember that, for the reasons given in the previous section, only random phenomena are considered; therefore, the RFVs have nil internal membership function (MF).

The considered methods are:

- the use of Yager's t-norm (in the following t-Yager), proposed by Urbanski in [10] and reproposed and applied in [11];
- the use of the product t-norm (in the following t-prod), proposed in [7];
- the used of the method proposed in [1, 9], based on the averaging operators (in the following RFV-method).

The following example is considered. Since only random phenomena are taken into account, let us start by considering two known probability distributions (pdf) *pA* and *pB*: a gaussian (with mean value $\mu = 1$ and standard deviation $\sigma = 0.1$) and a uniform pdf (with mean value $\mu = 2$ and a semi-amplitude of the support 0.5). These distributions are then transformed into the equivalent[1] MFs *A* and *B*, as defined in [1, 14]. Methods t-Yager, t-prod and RFV-method are then applied to *A* and *B* for the four elementary operations. This leads to Figs. 2-6.

In order to discuss the validity of the three different methods, the results obtained by applying Monte Carlo simulations are also drawn. In fact, pure random phenomena are considered and hence the Monte Carlo simulation is expected to provide a good approximation of the result.

In this respect, 10000 extractions[2] are taken from pdfs *pA* and *pB* and combined according to the considered operation; an histogram is build by considering 100 bins; a transformation from the histogram to the equivalent MF is done [1, 14].

However, it is important to consider that different Monte Carlo simulations generally provide different supports and different peak values of the histograms, and this could be a problem in the comparison of the three considered methods and the choice of the best one.

*SUPPORT.*

A Monte Carlo simulation combines randomly the extractions taken from the initial pdfs. From a strict theoretical point of view, only an infinite number of extractions

[1] The term equivalent, in this context, means that pdf *pA* and MF *A* define the same confidence intervals for all levels of confidence.

[2] The number of trials is a critical parameter in Monte Carlo simulations. In particular, Supplement1 [15] recommends to choose it adaptively. Anyway, in the considered case, a higher number of extractions did not show any improvement.

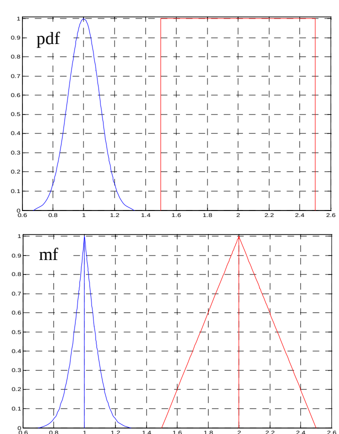


Fig. 2. Considered probability distributions and corresponding membership functions

can provide all the possible combinations between the values of the two pdfs. When a finite number of extractions is considered, only the most probable combinations are encountered. It follows that, whenever a Monte Carlo simulation is executed, the support of the obtained histogram is always smaller than the possible support of the result, which considers all possible combinations. Then, in different simulations, different supports are generally obtained. It follows that, when the probability-possibility transformation [1, 14] is applied, MFs with different supports are obtained, as shown, for instance, in Fig. 7. This problem cannot be eliminated and could be reduced only by generating a huge number of samples, making time consuming a critical aspect.

*PEAK VALUE.*

It is known that different Monte Carlo simulations provide different maximum values of the resulting pdf. It follows that, when the probability-possibility transformation [1, 14] is applied, MFs with different peak values are obtained, as shown, for instance, in Fig. 7. This makes the result obtained with Monte Carlo not comparable with the ones provided by the three considered methods, for which the peak value always coincide with the combination of the peak values of the initial MFs.

In order to overcome this last problem, in this paper, a slight modification of the probability-possibility transformation proposed in [1, 14] is followed: instead of considering the maximum value of the MF equal to the maximum value of the obtained histogram, the maximum value of the MF is forced to a known value, the expected value, given by the combination of the maximum values of the two initial pdfs[3]. This seems to be correct from a metrological point of view. In fact, if two measurement results *m* and *n* with 0 associated uncertainty are considered, the result of their combination is a number *P* with 0 associated uncertainty. If now, uncertainty intervals around the same measurement results *m* and *n* are considered, it is metrologically correct to assess that an interval (not necessarily symmetric) is added around *P*. Therefore, it seems to be correct, from a metrological point of view, to consider the combination of the maximum values of the two initial pdfs (*P*) as the peak value of the MF obtained through Monte Carlo.

From Figs. 3-6, the following considerations can be drawn.

**t-Yager**

The Yager's t-norm is not univocally defined, since its definition depends on a parameter *p*, with $p \geq 0$ [13]. The two dashed lines in Figs. 3-6 refers to values 0.5 and 5 (black and blue line respectively). Values lower than 0.5 provide solutions internal to the dashed black line; values greater than 5 provide solutions external to the dashed blue line; values between 0.5 and 5 provide solutions in between. It can be concluded that t-Yager should not be used for our metrological purposes, at least for two reasons. First of all, it provides a shape not as smooth as the shape of the Monte Carlo result (red line). Moreover, this t-norm always underestimates the uncertainty intervals at high levels of confidence, whichever is *p*. This can be seen by the abrupt narrowing of the $\alpha$-cuts at low $\alpha$ levels.

As a matter of fact, this is an intrinsic behavior of Yager's t-norm, since it is nilpotent [16]. As also clarified in section IIIA, nilpotent t-norms are not good for our purposes.

**t-prod**.

This t-norm provides (solid black line in Figs. 3-6) a very good approximation of the result when sum and difference are considered, but underestimates the uncertainty levels at all levels of confidence in the case of product and division. For this reason, also this method should be avoided for our purposes.

**RFV-method**.

The RFV-method (dashed-dotted black line in Figs. 3-6) is the only method proposed in the literature up to now that never underestimates the uncertainty intervals for any $\alpha$-level and for any operation.

## III. THE PROPOSAL: HAMACHER'S T-NORM

The above comparison between the available methods shows that methods t-Yager and t-prod are not able to represent the combination of measurement results affected by random contributions to uncertainty. On the contrary, the RFV-method can be used.

Nevertheless, Figs. 3-6 show that this method overestimates the uncertainty intervals for values $\alpha > 0$ [4].

It could be interesting to estimate whether and when this overestimation is acceptable.

If the measurement procedure is affected by all kinds of contributions to uncertainty and random contributions are small with respect to the other ones[5] [12, 17], the overestimation provided by the RFV-method is totally acceptable. Let us also consider that this method guarantees a very low time-consuming, as shown in next section C.

[3] Mean value in case of uniform pdfs.

[4] The provided support, given for $\alpha = 0$, is always correct [1].

[5] This is generally the case in most measurement procedures.

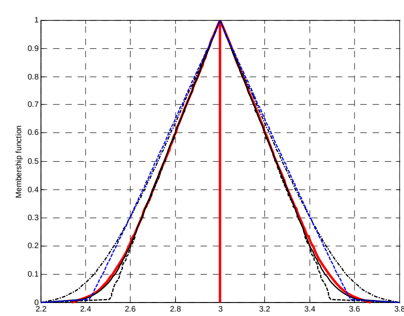


Fig. 3. Sum of *A* and *B*. Monte Carlo (red line), t-prod (solid line), Yager's t-norm (dashed line), RFV method (dashed-dotted line)

On the contrary, if the random contributions are comparable or greater than all non-random ones, it could be useful to achieve a better estimation of the result. In this section, a new method is proposed.

## A. Theoretical considerations

A triangular norm (t-norm) $T$ is a binary operation on the unit interval $[0, 1]$, that is commutative, associative, monotone and has 1 as the neutral element.

In particular, a continuous T-norm is said to be *Archimedean* if $T(x, x) < x$ for all $x \in (0, 1)$. A classical example of a continuous Archimedean T-norm is t-prod (i.e., $x^2 < x$ for $x \in (0, 1)$) whereas not Archimedean t-norms are t-min (i.e., $\min(x, x) = x$ for all $x \in (0, 1)$) and all t-norms built by ordinal sums using not Archimedean t-norm [16] (see Dombi t-norm for example). The Archimedean property is the basic assumption in many representation theorems used to implement t-norms by using alternative mathematical operators [16] (i.e., additive generators, multiplicative generators etc.). These different representations are needed in order to prove asymptotic properties of t-norms. As an example, consider the shape of the t-norm corresponding to the sum of $N$ variables as $N$ tends to infinity.

Moreover, a continuous Archimedean t-norm can be either *strict* or *nilpotent*. It is strict if 0 is the only nilpotent element such that $T(x, y) \neq 0$ for every $x, y \neq 0$ and $T(0, y) = T(x, 0) = 0$. On the contrary, a nilpotent T-norm is such that there exists at least $x \neq 0$ such that $T(x, y) = 0$ for every $y \neq 0$. An example of nilpotent t-norm is t-Yager for $p \in (0, \infty)$. In our context of uncertainty representation, nilpotence property cannot be accepted since there can be a nil confidence interval produced by combining nonzero $\alpha$-cuts. Typically, this fact causes an underestimation of the final $\alpha$-cuts for $\alpha$ close to zero (i.e., high confidence levels).

Hence, we look for a continuous strict Archimedean t-norm. t-norm t-Hamacher is proposed, defined as:

$$T_p^H(x, y) = \begin{cases} T_D(x, y), & p = \infty, \\ 0, & p = x = y = 0, \\ \frac{xy}{p + (1-p)(x + y - xy)}, & \text{otherwise.} \end{cases} \quad (1)$$

where $T_D$ is the drastic t-norm [16]. This t-norm is parametric and parameter $p \in [0, +\infty]$.

## B. Experimental results

Let us consider the same example taken into account in the previous section. Figs. 8-11 show the result obtained by applying Monte Carlo (red line), the RFV-method (dashed-dotted black line) and the Hamacher's t-norm (solid green line). In particular, a coefficient $p = 0.5$ is used for sum and difference, while $p = 0.1$ is used for product and division. The choice of the parameters has been done on the basis of the various simulations performed. The figures show the good approximation provided by the application of t-Hamacher.

## C. Implementation of Hamacher's t-norm on $\alpha$-cuts

Definition (1) is given in terms of membership functions. However, in order to avoid resolution and sampling problems [1], the best way to deal with RFVs is in terms of their $\alpha$-cuts. The use of this representation also allows to provide directly the $\alpha$-cuts of the final RFV, in other words, the confidence intervals at all levels of confidence [1].

Thanks to Nguyen theorem [18], it is possible to define each t-norm in terms of the $\alpha$-cuts of the two initial FVs *A* and *B*:

$$[f(A, B)]_\alpha = \bigcup_{T(\xi,\eta) \geq \alpha} f(A_\xi, B_\eta) \quad \alpha \in (0, 1] \quad (2)$$

where $T(\xi, \eta)$ is the considered t-norm, in our case, t-Hamacher (1). For $\alpha = 0$, $[f(A, B)]_\alpha = f(A_{\alpha=0}, B_{\alpha=0})$ is used.

However, the strict application of (2) is very long time consuming. In fact, (2) shows that the $\alpha$-cut at level $\alpha$ of the result of the operation on *A* and *B* is given by the union of a certain number of intervals $f(A_\xi, B_\eta)$, where $\xi$ and $\eta$ obey to $T(\xi, \eta) \geq \alpha$ and $A_\xi$ and $B_\eta$ are, respectively, the $\alpha$-cut at level $\xi$ of FV *A* and the $\alpha$-cut at level $\eta$ of FV *B*.

Hence, a faster method to implement (2) is given in this section. Let us first consider that, for every level $\alpha$, a certain number of couples $(\xi, \eta)$ satisfy $T(\xi, \eta) \geq \alpha$. It can be noted that the couples $(\xi, \eta)$ associated to the different levels $\alpha$ are univocally defined once the t-norm

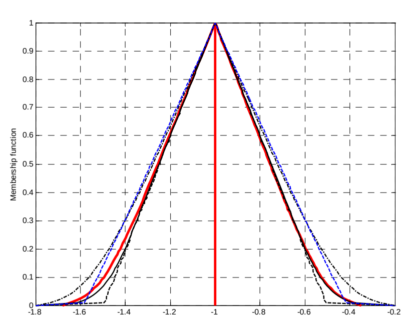


Fig. 4. Difference of *A* and *B*. Monte Carlo (red line), t-prod (solid line), Yager's t-norm (dashed line), RFV method (dashed-dotted line)

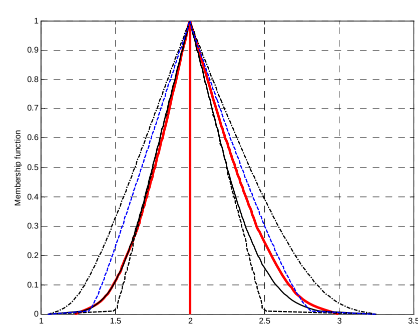


Fig. 5. Product of *A* and *B*. Monte Carlo (red line), t-prod (solid line), Yager's t-norm (dashed line), RFV method (dashed-dotted line)

and, if present, the parameter are chosen. Hence a matrix $M_1 = [\alpha, \xi, \eta]$ is univocally associated to a certain t-norm (for instance, t-Hamacher with $p = 0.5$). Moreover, studying at matrix $M_1$, it can be noted that, generally, more than one row contains the same values of $\alpha$ and $\xi$ and a different value of $\eta$. Since $f(A_\xi, B_{\eta_1}) \subset f(A_\xi, B_{\eta_2})$ if $\eta_2 > \eta_1$ and because of the union operator in (2), it follows that it is possible to consider only set of three $(\alpha, \xi, \eta_{min})$, where $\eta_{min}$ is the minimum value of $\eta$ associated to defined couples $(\alpha, \xi)$ in $M_1$.

Therefore, in order to make the implementation of (2) faster, it is possible to evaluate off-line the 3-column matrix $M = [\alpha, \xi, \eta_{min}]$ for the considered t-norm. Then, each time this t-norm is used, it is sufficient to recall matrix *M* and operate as follows. First, evaluate, for each level $\alpha$, intervals $f(A_\xi, B_\eta)$, where $\xi$ and $\eta$ are the numbers in the $2^{nd}$ and $3^{rd}$ columns of matrix $M$ for which the value in the $1^{st}$ column is $\alpha$; second, evaluate the union of all these intervals. With respect to the strict application of (2), the number of calculations is strongly reduced, thus reducing the total execution time.

If this procedure is followed, the following execution times are obtained in the case of Fig. 8:

- t-Hamacher: about 8.4s to evaluate matrix $M$ + about 0.2s for other calculations;
- RFV-method: about 25ms;
- Monte Carlo (10000 samples are generated for each variable): about 500ms;

showing that t-Hamacher is more time consuming. But, if the sum of a uniform, a gaussian and a triangular pdfs is considered (hence, two operations), it follows:

- t-Hamacher: $8.4s + 2 * 0.2s = 8.8s$;
- RFV-method: 78ms;
- Monte Carlo: 15.14s;

thus showing that the application of t-Hamacher becomes more and more convenient with respect to Monte Carlo as the number of operations arises. In fact, if matrices $M05$ (obtained by applying Hamacher with $p = 0.5$) and $M01$ (obtained by applying Hamacher with $p = 0.1$) are evaluated and stored, only the 0.2s time consuming for each operation can be considered.

## IV. CORRELATED RANDOM PHENOMENA

The random phenomena considered in the examples reported in this paper have been supposed to be uncorrelated. Indeed, the proposed mathematics is suitable only when uncorrelated random contributions to uncertainty must be composed. In case of partially or totally correlated random phenomena, a different approach must be followed. For the sake of brevity, it is impossible to study in depth these situations in the paper, that will be considered in future works.

## V. CONCLUSIONS

The paper shows that, between the different methods proposed in the literature for the combination of FV representing measurement results affected by only random contributions to uncertainty, only the RFV-method can be effectively employed. In fact, this method never underestimates the measurement uncertainty and is very efficient in terms of time consuming.

However, for $\alpha$-cuts for levels $\alpha > 0$, this method provides an overestimation of the confidence intervals. This overestimation is acceptable in most of the measurement procedure, for which the non-random effects are predominant over the random ones. For other situations and when it is necessary to estimate the final result better, Hamacher's t-norm is proposed in the paper. The application of this t-norm is more complicate than the RFV-method and requires more execution time. However, this is always less than the time required by the application of Monte Carlo whenever more than two operations have to be performed.

The use of t-Hamacher for the combination of RFVs is of a great importance, since it allows to combine measurement results referring exclusively to fuzzy operators and to the Theory of Evidence, with no reference to the probability theory.

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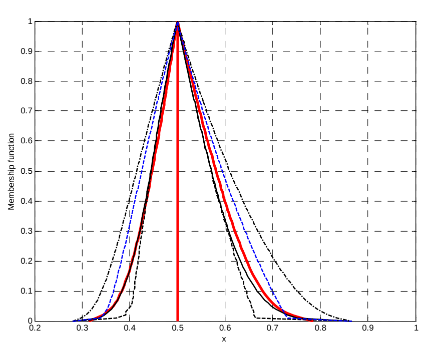


Fig. 6. Division of *A* and *B*. Monte Carlo (red line), t-prod (solid line), Yager's t-norm (dashed line), RFV method (dashed-dotted line)

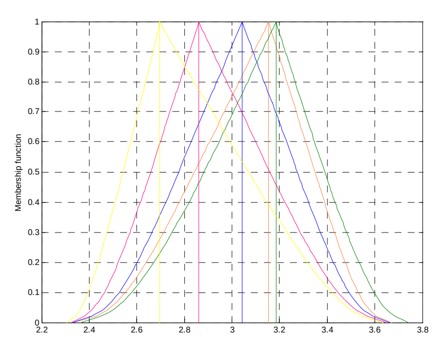


Fig. 7. Sum of *A* and *B*. Different simulations with Monte Carlo.

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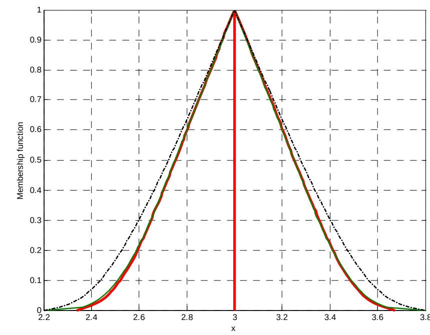


Fig. 8. Sum of $A$ and $B$. Monte Carlo (red line), RFV method (dashed-dotted black line), t-Hamacher (solid green line)

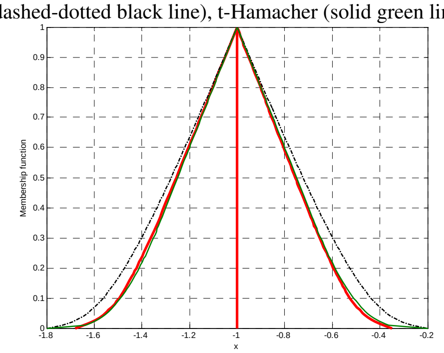


Fig. 9. Difference of $A$ and $B$. Monte Carlo (red line), RFV method (dashed-dotted black line), t-Hamacher (solid green line)

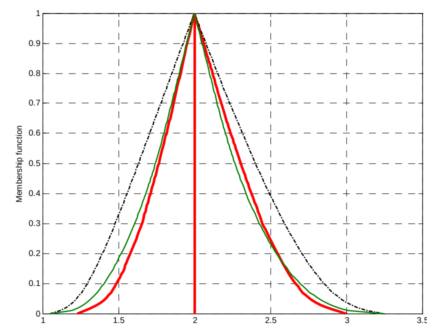


Fig. 10. Product of $A$ and $B$. Monte Carlo (red line), RFV method (dashed-dotted black line), t-Hamacher (solid green line)

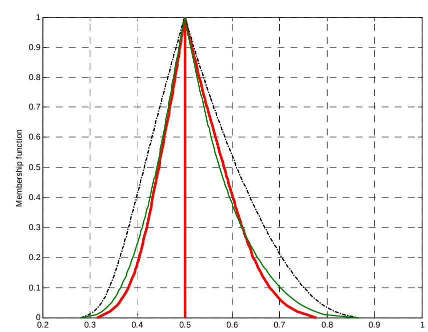


Fig. 11. Division of $A$ and $B$. Monte Carlo (red line), RFV method (dashed-dotted black line), t-Hamacher (solid green line)